Communication

1) Explain why the reciprocal of a polynomial shares the same points as the polynomial function at $y = \pm 1$.

For a polynomial $f(x)$, wherever it has $y$ values of $+1$ or $-1$, the reciprocal function $g(x) = \frac{1}{f(x)}$ will have values of $+1$ and $-1$ as well, because $\frac{1}{1} = 1$, and $\frac{1}{-1} = -1$. Therefore they share the same points.

2) When does a function have a hole? Give an example of a function with a hole and one without a hole.

A function has a hole when a factor completely cancels out of the denominator. (Note – the whole denominator doesn’t have to cancel! Just a factor!)

For example, the function $f(x) = \frac{(x+1)(x-2)}{(x+1)(x+5)}$ has a hole at $x = -1$ because the factor $(x + 1)$ cancels in the denominator.

If a factor does not fully cancel out of the denominator, then the function has a vertical asymptote. For example $f(x) = \frac{(x+1)(x-2)}{(x+1)^2(x+5)}$ will have an asymptote at $x = -1$.

3) Explain why a rational function with a higher degree in the numerator does not have a horizontal asymptote.

If a rational function has a higher degree in the numerator than the denominator, as the $x$ value goes to $\pm$ infinity, the numerator gets bigger faster than the denominator. So, the whole function goes towards $\pm$ infinity instead of getting closer and closer to a specific number.

4) Your friend thinks the function $f(x) = \frac{(x+2)(x+3)}{(x+3)^2}$ has a hole at $x = -3$ because $(x + 3)$ cancels out. Do you agree or disagree? Explain.

Disagree. The factor $(x + 3)$ must completely cancel out in the denominator or it there is a vertical asymptote at $x = -3$. 
5) Given \( f(x) = \frac{3x^3 + 2x^2 + 5x + 7}{x^2 + 2x} \), what is the oblique asymptote? Explain why you get an oblique asymptote using the division statement.

When you long divide, you get:

\[
\begin{array}{c|cc}
\text{Divide everything by } x^2 + 2x \\
\hline
x^2 + 2x & 3x^3 + 2x^2 + 5x + 7 \\
\hline
& R & 13x + 7 \\
\hline
\end{array}
\]

The division statement is \( 3x^3 + 2x^2 + 5x + 7 = (x^2 + 2x)(3x - 4) + 13x + 7 \)

Divide everything by \( x^2 + 2x \)

\[
\frac{3x^3 + 2x^2 + 5x + 7}{x^2 + 2x} = 3x - 4 + \frac{13x + 7}{x^2 + 2x}
\]

So the oblique asymptote is \( y = 3x - 4 \) because as \( x \to \infty \), \( \frac{13x + 7}{x^2 + 2x} \) goes to zero since the denominator has higher degree than the numerator.
1) **How many asymptotes can the reciprocal of a parabola have? Draw examples of each type.**

The reciprocal of a parabola can have 0, 1 or 2 asymptotes, because a parabola can have 0, 1, or 2 \( x \)-intercepts. The \( x \)-intercepts of the parabola create the vertical asymptotes of the reciprocal.
Thinking

1) Create two equations of rational functions with oblique asymptotes at $y = 2x - 3$.

$$y = 2x - 3 + \frac{7}{x - 1}$$

The above function would have an oblique asymptote at $y = 2x - 3$ because as $x$ goes to ± infinity, $\frac{7}{x-1}$ goes to zero.

$$y = \frac{(2x - 3)(x - 1) + 7}{x - 1}$$

To find the rational function, get a common denominator and simplify.

$$y = \frac{2x^2 - 5x + 3 + 7}{x - 1}$$

$$y = \frac{2x^2 - 5x + 10}{x - 1}$$

Another simpler way is to use an $x$ or $x^2$ in the denominator.

$$y = 2x - 3 + \frac{5}{x^2}$$

$$y = \frac{2x^3 - 3x^2 + 5}{x^2}$$

$$y = \frac{2x^3 - 3x^2 + 5}{x^2}$$

2) Create a rational function with $x$-intercept at 3 and $y$-intercept at $-1$.

For the function to have an $x$-intercept at 3, that means it has an $(x - 3)$ in the numerator. To have a $y$-intercept of $-1$, plugging in $x = 0$ must give a $-1$.

$$y = \frac{x - 3}{x + 3}$$

You could have many other functions:

$$y = \frac{(x - 3)(x + 2)}{x^2 + 6}$$
3) Create a rational function with a hole at $x = -5$, $x$-intercept at 2, $y$-intercept at $-\frac{1}{2}$, and horizontal asymptote at $y = \frac{1}{3}$.

$$y = \frac{x - 2}{x + 4}$$

This equation has an $x$-intercept at 2, because the numerator has a factor of $(x - 2)$. The $y$-intercept is $-\frac{2}{4}$ or $-\frac{1}{2}$ which is what we wanted. However the horizontal asymptote is currently $y = 1$.

To have a horizontal asymptote at $y = \frac{1}{3}$, you need the leading coefficient in the numerator to be 1, and the leading coefficient in the denominator to be 3. You also need the same degree in the numerator and the denominator.

$$y = \frac{x - 2}{3x + 4}$$

This equation has a horizontal asymptote at $y = \frac{1}{3}$ and we also have the correct $x$ and $y$ intercepts. We now just need to add a hole:

$$y = \frac{(x - 5)(x - 2)}{(x - 5)(3x + 4)}$$

To have a hole at $x = -5$, you need to have a factor $(x + 5)$ in the numerator and denominator so that the factor cancels in the denominator.

You could also have the equation:

$$y = \frac{(x - 5)^2(x - 2)}{(x - 5)^2(3x + 4)}$$
4) Create a function with oblique asymptote at \(-x + 4\), hole at \(x = 2\), vertical asymptote at \(x = -6\).

To get a vertical asymptote at \(x = -6\), the denominator must have an \((x + 6)\).

To get an oblique asymptote of \(y = -x + 4\), with the denominator \(x + 6\) we must have:

\[
y = -x + 4 + \frac{\text{some number}}{x + 6}
\]

Let the number be 5, but it will work with anything.

\[
y = -x + 4 + \frac{5}{x + 6}
\]

Get a common denominator.

\[
y = \frac{(-x + 4)(x + 6)}{x + 6} + \frac{5}{x + 6}
\]

\[
y = \frac{-x^2 - 2x + 24 + 5}{x + 6}
\]

\[
y = \frac{-x^2 - 2x + 29}{x + 6}
\]

To get a hole, just put in a factor of \((x - 2)\) in the numerator and denominator.

\[
y = \frac{(-x^2 - 2x + 29)(x - 2)}{(x + 6)(x - 2)}
\]